

Singular Calabi-Yau Manifolds and ADE Classification of CFTs

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Abstract

We study superstring propagations on the Calabi-Yau manifold which develops an isolated ADE singularity. This theory has been conjectured to have a holographic dual description in terms of $N = 2$ Landau-Ginzburg theory and Liouville theory. If the Landau-Ginzburg description precisely reflects the information of ADE singularity, the Landau-Ginzburg model of D_4, E_6, E_8 and Gepner model of $A_2 \otimes A_2, A_2 \otimes A_3, A_2 \otimes A_4$ should give the same result. We compute the elements of D_4, E_6, E_8 modular invariants for the singular Calabi-Yau compactification in terms of the spectral flow invariant orbits of the tensor product theories with the theta function which encodes the momentum mode of the Liouville theory. Furthermore we find the interesting identity among characters in minimal models at different levels. We give the complete proof for the identity.

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1 Introduction

Currently the study of superstring theory on singular Calabi-Yau manifolds is active. The important feature of superstrings propagating near singularities is the appearance of light solitons coming from the D-branes wrapped around the vanishing cycles. This is the non-perturbative quantum effect in string theory in the sense that even after taking $g_s \rightarrow 0$, the VEV of dilaton will blow up at the singular point. In order to appear such an effect, the vanishing worldsheet theta angle is necessary [1], which seems to make worldsheet CFTs singular [2]. In contrast to the well-established perturbative description of smooth Calabi-Yau manifolds like Gepner models [3], the worldsheet description of singular Calabi-Yau manifolds remains to be investigated. This situation is expected to bring us the new source of insights of stringy dynamics related to the space-time singularity. Moreover, this set up only depends on the type of the singularity, and has an interesting physical application that the decoupled theory in this background corresponds to, for example, four-dimensional CFTs classified by ADE [4].

Such a CFT can be described by the $N = 2$ Landau-Ginzburg model with a superpotential including a negative power of some chiral superfield in order to push up the central charge to the right value for Calabi-Yau manifolds, and this peculiar term was handled by Kazama-Suzuki model [5] for the non-compact coset $SL(2, \mathbf{R})/U(1)$ [6, 7]. Recently, after the renowned AdS/CFT correspondence [8], the approach based on a holographic point of view was proposed [9]. In this approach, the sector of Landau-Ginzburg theory with a negative power superpotential is replaced by Liouville theory [10]. This Liouville field corresponds to an extra non-compact direction, which indicates holography.

For this description, the first consistency check is to make the modular invariant partition function on a torus, and to see that the partition function vanishes. This issue was systematically pursued in [7, 11] for the singular Calabi-Yau manifold with an isolated ADE singularity (or conifold [12]). Furthermore, the extension to more complicated singularities was made [13]. In all the case, the string theory on the Calabi-Yau manifold with the ADE singularity is treated as $N = 2$ minimal model classified by ADE and Liouville theory.

Now, we wish to pose the point of view taken throughout this paper. In the Landau-Ginzburg description of $N = 2$ minimal models, the following relation should hold [14]:

$$D_4 = A_2 \otimes A_2, \quad E_6 = A_2 \otimes A_3, \quad E_8 = A_2 \otimes A_4, \quad (1)$$

with the appropriate projection in the CFTs. This is because D_4, E_6, E_8 simple singularities are defined by $x^3 + xy^2 = 0$, $x^3 + y^4 = 0$, $x^3 + y^5 = 0$, up to quadratic terms. We can use Gepner's

construction [3] for the tensor product theory in order to reproduce the feature of block diagonal D_4, E_6, E_8 modular invariants. At first sight, one may think that D_4 invariant is not written by the tensor product theory due to the term xy^2 in the above polynomial. However we can rewrite the polynomial into the form $x^3 + y^3$ by the suitable linear transformation of variables. Above relation (1) can be reduced to the identity among characters in minimal models at different levels. In principle, this can be predicted by the comparison of weight and $U(1)$ charge of the irreducible representations in minimal models. But the physical meaning has been lacking. The partition function for singular Calabi-Yau manifolds is the slight generalization of well-known ADE classification of modular invariant (for the recent discussion, see [15]). If this partition function reflects the singularity of spacetime, there should exist the reformulation of D_4, E_6, E_8 invariants by the tensor product theory as in (1) and the identity among characters in minimal models. This is our interpretation of the conventional simple singularity. However, we cannot proceed the other non-diagonal modular invariants $D_{n>4}, E_7$.

In order to make modular invariant of the tensor product theory, we use the so-called spectral flow method [16] which makes the block diagonalization and the spacetime supersymmetry manifest. One reason for using the spectral flow method is that for singular K3 surface, we can reproduce the block diagonal elements of D_4, E_6, E_8 modular invariants as the spectral flow invariant orbits by the tensor product theory of suitable minimal models. This cannot be seen only with the minimal models. Due to the identity among the characters of minimal models, the extension to the singular Calabi-Yau 3 or 4-folds is straightforward. Finally we give the proof of the proposed identity.

The paper is organized as follows. In section 2, we briefly review the construction of the modular invariant partition function in the system of superstrings on Calabi-Yau manifolds with the isolated ADE singularity, based on the Liouville system and minimal model classified by ADE. Section 3 is devoted to introduce the necessary background and notation about the spectral flow method used in the remainder of the paper. In section 4, we propose our main results about the D_4, E_6, E_8 modular invariants. We explicitly identify the block diagonal elements of D_4, E_6, E_8 invariants by the spectral flow invariant orbits of Gepner models of $A_2 \otimes A_2, A_2 \otimes A_3, A_2 \otimes A_4$ and derive the identity among characters in minimal models at different levels. Section 5 includes the conclusion and discussion. In the Appendix A, we collect the formula about the theta functions and characters of minimal models, used in this paper extensively. Appendix B includes the exact proof for the identity among the characters in minimal models.

2 Superstrings on singular Calabi-Yau manifolds

In this section, we briefly review the construction of the modular invariant partition function of non-critical superstring theory which is conjectured to give the dual description of the Calabi-Yau manifolds with ADE singularity in the decoupling limit [7, 11].

2.1 CFT for non-critical superstring

Let us consider Type II string theory on the background $\mathbf{R}^{d-1,1} \times X_n$, where X_n is a Calabi-Yau n -fold ($2n + d = 10$) with an isolated singularity, locally defined by $F(x_1, \dots, x_{n+1}) = 0$ in appropriate weighted projective space. In particular, we concentrate on the isolated rational ADE singularity. For singular 2-fold, or K3 surface, the following polynomials define the singular geometries

$$\begin{aligned} F_{A_{N-1}} &= x_1^N + x_2^2 + x_3^2, & (N \geq 2) \\ F_{D_{\frac{N}{2}+1}} &= x_1^{N/2} + x_1 x_2^2 + x_3^2, & (N : \text{even} \geq 6) \\ F_{E_6} &= x_1^4 + x_2^3 + x_3^2, \\ F_{E_7} &= x_1^3 x_2 + x_2^3 + x_3^2, \\ F_{E_8} &= x_1^5 + x_2^3 + x_3^2. \end{aligned} \tag{2}$$

For singular Calabi-Yau 3,4-folds, we add $x_4^2, x_4^2 + x_5^2$ to above polynomials. The quadratic terms do not change the type of singularity.

In the decoupling limit $g_s \rightarrow 0$, we obtain a non-gravitational, and maybe non-trivial quantum theory on $\mathbf{R}^{d-1,1}$. These d -dimensional quantum theories are expected to flow into non-trivial conformal RG fixed points in the IR limit.

According to the holographic duality [10], we have the dual description of the above system in terms of non-critical superstrings on

$$\mathbf{R}^{d-1,1} \times (\mathbf{R}_\phi \times S^1) \times LG(W = F), \tag{3}$$

where $LG(W = F)$ denotes the $N = 2$ Landau-Ginzburg model with a superpotential $W = F$. And \mathbf{R}_ϕ denotes a linear dilaton background with the background charge $Q(> 0)$. The part $\mathbf{R}_\phi \times S^1$ is described by the $N = 2$ Liouville theory [17] whose matter content consists of bosonic fields ϕ, Y which parameterize \mathbf{R}_ϕ, S^1 , respectively, and their fermionic partners ψ^+, ψ^- . Then

the $N = 2$ superconformal currents are written as

$$\begin{aligned} T &= -\frac{1}{2}(\partial Y)^2 - \frac{1}{2}(\partial\phi)^2 - \frac{Q}{2}\partial^2\phi - \frac{1}{2}(\psi^+\partial\psi^+ - \partial\psi^+\psi^-), \\ G^\pm &= -\frac{1}{\sqrt{2}}\psi^\pm(i\partial Y \pm \partial\phi) \mp \frac{Q}{\sqrt{2}}\partial\psi^\pm, \\ J &= \psi^+\psi^- - Q i\partial Y, \end{aligned} \tag{4}$$

which generate the $N = 2$ superconformal algebra with central charge $c = 3 + 3Q^2$.

Here we consider a linear dilaton background, so the string theory is weakly coupled in the region far from the singularity. On the other hand, the string coupling constant diverges near the singularity, hence the perturbative approach is not reliable. Thus we must add the Liouville potential to the worldsheet action of the Liouville theory in order to guarantee that strings do not propagate into the region near the singularity. But this additional term is actually the screening charge which commutes with all the generators of $N = 2$ superconformal algebra (3). Thus although we cannot set the actual interaction to be vanish, we can pursue all the manipulations like a free worldsheet CFT without the Liouville potential. This situation is physically realized by taking the double scaling limit in [18]. This limit holds the mass of wrapped branes at finite value, so we may not see the gauge symmetry enhancement, which is characteristic phenomena at the singularity.

For the isolated ADE singularity, the Landau-Ginzburg theory with $W = F$ is nothing but the $N = 2$ minimal models (MM) classified by ADE, and have the central charge $c = \frac{3(N-2)}{N}$, where $N = k + 2$ is the dual Coxeter number of the ADE groups. In this paper, we use both N and k in order to show the level of $N = 2$ minimal model, however it may be no confusion.

The condition for cancellation of conformal anomaly can be written as

$$d \times \left(1 + \frac{1}{2}\right) + \frac{3N-6}{N} + 3(1+Q^2) + 11 - 26 = 0, \tag{5}$$

and then it is easy to determine the background charge Q for each of the cases $d = 6, 4, 2$. In the case $d = 6$, we obtain

$$Q = \sqrt{\frac{2}{N}}. \tag{6}$$

For the case of singular K3 surface, the corresponding Landau-Ginzburg theories are described by the following superpotential [7]

$$W_G = z^{-N} + F_G, \tag{7}$$

where $G = ADE$ and F_G is defined by (2). These non-compact Landau-Ginzburg theories describe conformal field theories with $c = 6$, which can be reinterpreted by the coset models

$$\left(\frac{SL(2)_{N+2}}{U(1)} \times \frac{SU(2)_{N-2}}{U(1)} \right) / \mathbf{Z}_N. \quad (8)$$

The non-compact z -dependent piece, corresponding to the $SL(2)$ factor in the coset plays a role to push up the central charge into the right value. The equivalence between this non-compact Kazama-Suzuki model and the $N = 2$ Liouville theory was discussed in [7, 11]. It was pointed out that both theories are related by a kind of T -duality [7].

2.2 Modular invariant partition function on a torus

Let us consider the modular invariant partition function on a torus for the above non-critical superstrings in the light-cone gauge $(\mathbf{R}^{d-2} \times \mathbf{R}_\phi \times S^1 \times MM)$. The toroidal partition function factorizes into two parts

$$Z_0(\tau, \bar{\tau}) Z_{GSO}(\tau, \bar{\tau}), \quad (9)$$

where Z_{GSO} contains the contributions on which GSO projection acts non-trivially, and we denote the remaining part by Z_0 .

The part Z_0 has only the contributions from the transverse non-compact bosonic coordinates $\mathbf{R}^{d-2} \times \mathbf{R}_\phi$, or the flat spacetime bosonic coordinates and the linear dilaton ϕ . The Liouville sector is a bit subtle because of the background charge. We use the ansatz that only the normalizable states contribute to the partition function. The normalizable spectrum in Liouville theory, in the sense of the delta function normalization because the spectrum is continuous, has the lower bound $h = Q^2/8$ [19]. This bound is nonzero, thus we must carefully handle the integration over the zero-mode momentum. However it turns out that the resulting partition function of ϕ is effectively the same as that of an ordinary boson because the effective value of the Liouville central charge $c_{\text{eff},L}$ is equal to

$$c_{\text{eff},L} \equiv (1 + 3Q^2) - 24 \times \frac{Q^2}{8} = 1, \quad (10)$$

which is independent of the background charge [19]. Note that it is not clear whether we should include the other modes. However we do not concern with that point in this paper.

Thus we obtain Z_0 effectively as the contribution from $d - 1$ free bosons

$$Z_0(\tau, \bar{\tau}) = \left(\frac{1}{\sqrt{\tau_2} |\eta(\tau)|^2} \right)^{d-1}, \quad \tau = \tau_1 + i\tau_2, \quad (11)$$

which is manifestly modular invariant.

The part Z_{GSO} should be treated separately for $d = 6, 4, 2$ due to the specific GSO projection. We only mention the simplest case $d = 6$, corresponding to singular K3 surface.

In order to specify the GSO projection, we have to consider the Fock space of the bosonic circular space-time coordinates Y constructed on the Fock vacuum $|p\rangle$, $\oint i\partial Y|p\rangle = p|p\rangle$. The values of the momenta p are chosen in consistent with the GSO projection. The conditions for the GSO projection on the $U(1)$ charge, which ensures the mutual locality with the space-time SUSY charges, are given by the following manner [10]

$$\begin{aligned} F + F_{MM} + \frac{m}{N} - pQ &\in 2\mathbf{Z} + 1, & \text{NS sector,} \\ F + F_{MM} + \frac{m}{N} - pQ &\in 2\mathbf{Z}, & \text{R sector,} \end{aligned} \quad (12)$$

where F denotes the fermion number of $\mathbf{R}^{d-2} \times (\mathbf{R}_\phi \times S^1)$ sector and F_{MM} denotes the fermion number of the minimal model (For the notation of minimal model, see the Appendix A).

We compute the trace over the left-moving Hilbert space. For example, consider the NS sector with $F + F_{MM} \in 2\mathbf{Z} + 1$. The sum over the momenta becomes

$$\sum q^{\frac{1}{2}p^2} = \sum_n q^{\frac{N}{4}(2n+\frac{m}{N})^2} = \sum_n q^{N(n+\frac{m}{2N})^2} = \theta_{m,N}(\tau). \quad (13)$$

Then with the factors coming from oscillator modes and the minimal modes, we obtain

$$\frac{1}{2} \left[\left(\frac{\theta_3}{\eta} \right)^3 \text{Ch}_{\ell,m}^{NS,(N-2)} + \left(\frac{\theta_4}{\eta} \right)^3 \widetilde{\text{Ch}}_{\ell,m}^{NS,(N-2)} \right] \frac{\theta_{m,N}}{\eta}, \quad (14)$$

where η, θ_i ($i = 2, 3, 4$) are the usual Dedekind, Jacobi theta functions. In this contribution, $\left(\frac{\theta_i}{\eta}\right)^2$ comes from the $S\widehat{O}(4)_1$ character which is the contribution of the fermionic fields in \mathbf{R}^4 . Additional contribution $\frac{\theta_i}{\eta}$ comes from the contribution of fermionic fields in $N = 2$ Liouville theory. Almost in the same way, we can write down the whole contribution by

$$\begin{aligned} \frac{1}{2} \sum_{\ell=0}^{N-2} \sum_{m \in \mathbf{Z}_{2N}} &\left[\theta_3^3 \text{Ch}_{\ell,m}^{NS,(N-2)} (\theta_{m,N} + \theta_{m+N,N}) - \theta_4^3 \widetilde{\text{Ch}}_{\ell,m}^{NS,(N-2)} (\theta_{m,N} - \theta_{m+N,N}) \right. \\ &\left. - \theta_2^3 \text{Ch}_{\ell,m}^{R,(N-2)} (\theta_{m,N} + \theta_{m+N,N}) \right], \end{aligned} \quad (15)$$

where \widetilde{R} sector vanishes, and we have omitted the factor of η function for simplicity. Note that this sum counts each state twice due to the field identification for the character of minimal model. In order to avoid this double counting, it is convenient to define

$$F_\ell(\tau) \equiv \frac{1}{2} \sum_{m \in \mathbf{Z}_{2N}} \theta_{m,N} \left(\theta_3^3 \text{Ch}_{\ell,m}^{NS,(N-2)} - \theta_4^3 \widetilde{\text{Ch}}_{\ell,m}^{NS,(N-2)} - \theta_2^3 \text{Ch}_{\ell,m}^{R,(N-2)} \right) (\tau), \quad (16)$$

and construct modular invariants using this F_ℓ .

Although we can read off the modular property of F_ℓ directly from the above definition, it is convenient to introduce F_ℓ with z dependence

$$F_\ell(\tau, z) = \frac{1}{2} \sum_{m \in \mathbf{Z}_{2N}} \theta_{m,N}(\tau, -2z/N) \times \left(\theta_3^3 \text{Ch}_{\ell,m}^{NS,(N-2)} - \theta_4^3 \widehat{\text{Ch}}_{\ell,m}^{NS,(N-2)} - \theta_2^3 \text{Ch}_{\ell,m}^{R,(N-2)} - i\theta_1^3 \widehat{\text{Ch}}_{\ell,m}^{R,(N-2)} \right) (\tau, z). \quad (17)$$

Then due to the branching relation (77), we can express F_ℓ in the following form [7]

$$F_\ell(\tau, z) = \frac{1}{2} \left(\theta_3^4 - \theta_4^4 - \theta_2^4 + \theta_1^4 \right) (\tau, z) \chi_\ell^{(N-2)}(\tau, 0), \quad (18)$$

where $\chi_\ell^{(k)}$ denotes $\widehat{SU(2)}_k$ character of the spin $\ell/2$ representation (76). Thanks to this relation, we can easily find that F_ℓ shows the same modular transformation property as the affine $SU(2)$ character. Now we can construct the modular invariant partition function on a torus

$$Z_{GSO}(\tau, \bar{\tau}) = \frac{1}{|\eta(\tau)|^8} \sum_{\ell, \bar{\ell}=0}^{N-2} N_{\ell, \bar{\ell}} F_\ell(\tau) F_{\bar{\ell}}(\bar{\tau}), \quad (19)$$

where $N_{\ell, \bar{\ell}}$ is the Cappelli-Itzykson-Zuber matrix, which can be classified by ADE [20]. In this way, we can obtain the modular invariant classified by the ADE groups corresponding to the singularity type of X_n [7, 21]. In this partition function, it appears a mass gap and the continuous spectrum above the gap due to the Liouville theory.

Note that F_ℓ identically vanishes by virtue of the Jacobi's abstruse identity in (18). This is consistent with the existence of space-time supersymmetry. Furthermore, the appearance of the $\widehat{SU(2)}$ character in (18), and the standard ADE classification of modular invariant corresponding to the type of degeneration of K3 surface, which coincides exactly with the well-known modular invariants of $SU(2)$ WZW theory, are quite satisfactory pictures. This originates from the following argument. In some sense, we can relate the background of singular K3 surface to a collection of NS5-branes by means of T -duality [7]. Moreover, it was argued that the world-sheet CFT of superstrings on NS5 brane background contains the $SU(2)$ WZW theory in the near horizon regime [22].

In the case of singular three- or four-fold with an isolated ADE singularity, we can similarly construct modular invariant partition function on a torus using the ADE classification of modular invariant [11]. However the worldsheet interpretation of the results is not so much clear as the singular K3 surface. Except for the conifold, we do not know the dual description by intersecting NS5 branes [23].

3 Gepner models

In this section we review the construction of modular invariant partition function of Gepner models by the spectral flow method [16]. A characteristic feature of this method is that the space-time supersymmetry is manifest and the partition function on torus can be constructed by the block diagonal way. As we will see in the next section, this block diagonal partition function is essential in order to see the proposed identifications.

We first mention the original work of Gepner [3] and then give a brief review of spectral flow method. We only consider the smooth Calabi-Yau compactifications in this sections. However the basic idea does not change even though one consider Calabi-Yau manifolds with isolated ADE singularities.

3.1 Spectral flow method

Let us discuss type II string theory compactified on the smooth Calabi-Yau manifolds of complex dimension n ($n = 1$ for the torus, 2 for the K3 surface and 3 for the Calabi-Yau threefold). The transverse space in the light-cone gauge is described by free bosons and free fermions. To describe the internal space by exactly solvable CFT, one consider a tensor product of r $N = 2$ minimal models of level k_1, \dots, k_r . In fact, we need certain conditions to construct the supersymmetric string compactifications. The cancellation of the trace anomaly requires that the central charges of minimal models must add up to $3n$. We further impose the projection that total $U(1)$ charges (sum of the charges from the transverse SCFT and from the internal SCFT) in both the left moving and right moving sector should be odd integers. Then we require the sector arraignment, which means that the left-moving states and the right-moving states must be taken from the NS sectors of each sub-theory or from the R sectors of each sub-theory and do not mix both sectors. Gepner [3] constructed the consistent modular invariant partition functions which is compatible with these conditions. This is so-called the β -method. However, the result is too complicated and block diagonalization of partition function can not be seen manifestly, so this procedure is not suitable for our goal. Thus we give another so-called spectral flow method, which gives the same result as the β -method.

A well-known feature of the $N = 2$ algebra is the isomorphism of the algebra under the continuous shift of the moding of the generators, i.e. under the spectral flow,

$$L_n \rightarrow L_n + \eta J_n + \frac{1}{6} c \eta^2 \delta_{n,0},$$

$$\begin{aligned} J_n &\rightarrow J_n + \frac{1}{3} c \eta \delta_{n,0}, \\ G_r^\pm &\rightarrow G_{r\pm\eta}^\pm, \end{aligned} \quad (20)$$

where L_n , J_n , G_r^\pm are Virasoro, $U(1)$ current and supercharge generators, respectively and η is an arbitrary real parameter. The space-time supersymmetry transformation corresponds to the shift $\eta \rightarrow \eta + \frac{1}{2}$, which exchanges NS sector for R sector. Thus if the total Hilbert space is invariant under the shift $\eta \rightarrow \eta + \frac{1}{2}$, the supersymmetry is manifest. Further under the shift $\eta \rightarrow \eta + 1$, NS sector comes back to NS sector, however in general the states in NS sector are mapped onto the different states. Therefore we repeat to operate the spectral flow until we return to the original states.

Partition function of Gepner models on which GSO projection acts non-trivially is expressed in terms of the characters of $N = 2$ minimal model and free fermion. For a given representation of $N = 2$ minimal model, we can define the characters in each sector (see Appendix A). Under the spectral flow with parameter $\eta = \frac{1}{2}$ or equivalently $z \rightarrow z + \frac{\tau}{2}$ with a factor $q^{\frac{c}{24}} y^{\frac{c}{6}}$, which comes from the shift of zero mode in (20), the character in the NS sector becomes

$$q^{\frac{c}{24}} y^{\frac{c}{6}} \text{Ch}_{\ell,m}^{NS,(k)} \left(\tau, z + \frac{\tau}{2} \right) = \text{Ch}_{\ell,m-1}^{R,(k)}(\tau, z) \quad (21)$$

and under the full shift $\eta = 1$ or $z \rightarrow z + \tau$ with a factor $q^{\frac{c}{6}} y^{\frac{c}{3}}$, the character in the NS sector becomes

$$q^{\frac{c}{6}} y^{\frac{c}{3}} \text{Ch}_{\ell,m}^{NS,(k)}(\tau, z + \tau) = \text{Ch}_{\ell,m-2}^{NS,(k)}(\tau, z), \quad (22)$$

where we have the same expression in R sector.

Let us consider how to construct the partition function of Gepner models by the spectral flow method. We first define “supersymmetric characters” which is the building block of the partition function. We multiply all the characters in NS sector which include the ground state $h = q = 0$, i.e. $\text{Ch}_{0,0}^{NS,(k_1)} \dots \text{Ch}_{0,0}^{NS,(k_r)}$. Then we apply the $\eta = 1$ spectral flow operations (22) until we obtain the original state. We denote these spectral flow invariant combination by NS_0 . The graviton corresponds to $h = 0$ state and NS_0 is called ‘graviton orbit’. Under the modular transformations $S : \tau \rightarrow -\frac{1}{\tau}$, we obtain a family of new spectral flow invariant orbits NS_i (the range of i depends on the models). We iterate this procedure until they transform among themselves under the modular S transformation:

$$NS_i \left(-\frac{1}{\tau} \right) = \sum_j S_{ij} NS_j(\tau), \quad (23)$$

where S_{ij} is real S -matrix satisfying $S^2 = 1$.

Then the contribution from other sectors is obtained in a straightforward way. The orbits in the R sector can be obtained by the spectral flow (21) and the orbits in the $\widetilde{\text{NS}}$ or the $\widetilde{\text{R}}$ sector, which are needed to close the orbits under the modular transformations $S : \tau \rightarrow -\frac{1}{\tau}$ and $T : \tau \rightarrow \tau + 1$, are obtained by the flow $z \rightarrow z + \frac{1}{2}$. The modular transformation matrix of these orbits is the same as that of (23) [16]. Therefore we introduce the supersymmetric character

$$X_i(\tau, z) = \frac{1}{2} \left\{ NS_i \left(\frac{\theta_3}{\eta} \right)^m - \widetilde{NS}_i \left(\frac{\theta_4}{\eta} \right)^m - R_i \left(\frac{\theta_2}{\eta} \right)^m + \widetilde{R}_i \left(\frac{\theta_1}{\eta} \right)^m \right\} (\tau, z), \quad (24)$$

where $(\frac{\theta}{\eta})^m$ come from the $SO(\widehat{2m})_1$ characters with $m = 4 - n$ which is the contribution of the spinor fields of the transverse flat space. This character is spectral flow invariant and therefore space-time supersymmetry is manifest.

Now we would like to construct the modular invariant partition function on a torus. Under the modular T transformation, the supersymmetric character is invariant up to a total phase factor. Under the modular S transformation, S -matrix of the character is identical to that of NS_i (23). Therefore we can easily construct the modular invariant partition function. We can define a particular diagonal matrix D

$$D_i = \frac{S_{0i}}{S_{i0}}, \quad (25)$$

satisfying

$$\sum_i S_{ij} D_i S_{ik} = D_j \delta_{jk}. \quad (26)$$

Then the partition function on the torus is obtained by the following bilinear modular invariant combination

$$Z = \sum_i D_i |X_i|^2. \quad (27)$$

We can check that this gives the same partition function constructed by the β -method. A characteristic feature of the spectral flow method is that if we want to construct the supersymmetric characters and to know the modular transformation of them, we have only to obtain the flow invariant orbits of the NS sector and the modular invariance among themselves.

Finally we should comment on the reason for the block diagonalization of partition function. In general, if the theory has a certain enlarged algebra in the theory, partition function is block diagonalized or fully diagonalized in that algebra. If we include the generators of spectral flow with $\eta = \pm 1$ to the original $N = 2$ algebra, we can extend it to enlarged algebra (in particular, in the case of K3 surface the algebra becomes $N = 4$ superconformal algebra). Thus in this enlarged algebra, the partition function is block diagonalized or fully diagonalized.

4 D_4, E_6, E_8 modular invariants from tensor products

Let us consider the $N = 2$ Landau-Ginzburg theory in two dimensions [26]. Due to the singularity theory, the form of the superpotential is classified by ADE, which correspond to $c < 3$ unitary $N = 2$ conformal minimal models which have the same ADE classification as $SU(2)$ WZW models [20]. The validity of this picture is checked by the equivalence of elliptic genus [27, 28]. In particular, we concentrate on the D_4, E_6, E_8 modular invariants. These are very special modular invariants only with the block diagonal form, which may signal that the representations in these partition functions form the reducible representation of original $\widehat{SU}(2)$ symmetry. Originally we have a whole Hilbert space spanned by all the states with $\ell = 0, \dots, k$ (spin $\ell/2$ representations), but the solutions tell us that there exist sensible physical system which have only the exponent of D_4, E_6, E_8 groups. In the case of D_4 , the appearance of factor 2 is the specific feature. Also, in E_6, E_8 case, we can expect that combined with some larger symmetry, the partition function will be diagonalized [24]. However this larger symmetry hides the original relation between $SU(2)$ WZW models and $N = 2$ minimal model by GKO coset construction [25].

In view of Landau-Ginzburg potential (2), we can expect that D_4, E_6, E_8 theories can be recaptured via $A_2 \otimes A_2, A_2 \otimes A_3, A_2 \otimes A_4$ Gepner models [14]. For D_4 case, we can rewrite the polynomial into the form $x^3 + y^3$. We should be able to see this correspondence at the level of the partition functions. However we can not make modular invariant partition function using spectral flow invariants only with $A_2 \otimes A_2, A_2 \otimes A_3, A_2 \otimes A_4$ minimal models. If we try to construct the spectral flow orbit only with minimal models, we encounter the bad T -transformation property due to the absence of integrality of $U(1)$ charge. Of course, also in the case of smooth Calabi-Yau compactification classified by ADE [29], we can expect above phenomena. But due to the complexity, this problem has not been investigated.

Thus we wish to consider this identification in the singular Calabi-Yau compactification. In the present situation, we have originally have the negative power superpotential, which is somewhat difficult to tackle. But we have replaced the negative term with the Liouville field, so we may use the above standard logic. Moreover as you can see from previous sections, the spectral flow method is quite more suitable with this situation than the β -method in order to construct the tensor product theory. Furthermore, we can construct the explicit space-time supersymmetric multiplet. Here, we have the following two interests. At first, we have an interest to reveal peculiar Hilbert space contained in block diagonal D_4, E_6, E_8 invariants. In the second, this is the simplest setting to make the tensor product theory. Our interest

is the correspondence between these two objects. Moreover this is the necessary consistency check if the partition function reflects the singularity in spacetime. We can make the similar construction for more complicated singularity in [13], but it is straightforward and may not be meaningful for the purpose in this paper.

In the off-diagonal cases of D_5, E_7 , the Landau-Ginzburg potentials are quartic. Then we can rewrite the potential naively as in D_4 case. However we cannot make the tensor product theory with the $c = 9/4, 8/3$, which corresponds to the value of central charge of D_5, E_7 theory. For $D_{>6}$ case, we cannot rewrite the potential as in D_4 case. Thus the other modular invariants $D_{n>4}, E_7$ seem not to be constructed by the tensor product theory.

4.1 D_4 case

We wish to reproduce the D_4 modular invariant in (19)

$$Z_{GSO}^{D_4} = \frac{1}{|\eta|^8} \left(|F_0 + F_4|^2 + 2|F_2|^2 \right), \quad (28)$$

by the modular invariant of $A_2 \otimes A_2$ Gepner model.

In the NS sector of A_2 minimal model at level 1, there are three irreducible representations. We denote the characters associated with these representations as

$$A_1 = \text{Ch}_{0,0}^{NS,(1)}, \quad B_1 = \text{Ch}_{1,1}^{NS,(1)}, \quad C_1 = \text{Ch}_{1,-1}^{NS,(1)}. \quad (29)$$

Under the spectral flow with $\eta = 1$, the above characters change as follows

$$A_1 \rightarrow B_1 \rightarrow C_1 \rightarrow A_1, \quad (30)$$

which is the diagrammatic expression of the flow (91).

In order to construct the modular invariant partition function, we have to specify the condition of GSO projection. The GSO projection is given by

$$\begin{aligned} F + F_{MM_1} + F_{MM_2} + \frac{m_1 + m_2}{3} - \frac{p}{\sqrt{3}} &\in 2\mathbf{Z} + 1, & \text{NS sector,} \\ F + F_{MM_1} + F_{MM_2} + \frac{m_1 + m_2}{3} - \frac{p}{\sqrt{3}} &\in 2\mathbf{Z}, & \text{R sector,} \end{aligned} \quad (31)$$

which is obvious generalization of (12) and MM_1, MM_2 represent two A_2 minimal models. Then let us calculate the trace over the left-moving Hilbert space. At first, consider the NS

sector with $F + F_{MM_1} + F_{MM_2} \in 2\mathbf{Z} + 1$. The sum over the momenta becomes

$$\sum_n q^{\frac{1}{2}p^2} = \sum_n q^{\frac{3}{2}(2n+\frac{m_1+m_2}{3})^2} = \sum_n q^{6(n+\frac{m_1+m_2}{6})^2} = \theta_{2m_1+2m_2,6}(\tau). \quad (32)$$

On the other hand, in the NS sector with $F + F_{MM_1} + F_{MM_2} \in 2\mathbf{Z}$, we obtain the following sum

$$\sum_n q^{\frac{1}{2}p^2} = \sum_n q^{\frac{3}{2}(2n+1+\frac{m_1+m_2}{3})^2} = \sum_n q^{6(n+\frac{m_1+m_2+3}{6})^2} = \theta_{2m_1+2m_2+6,6}(\tau). \quad (33)$$

We have to make the spectral flow invariant orbit and the modular invariant of the tensor product theory, then identify the pieces which coincide with the block diagonal elements of the D_4 modular invariants (28). In order to make the orbit in the manner as section 3, we adopt the simple ansatz that the graviton orbit contains the term

$$A_1^2(\tau, z) \theta_{0,6}(\tau, -z/3), \quad (34)$$

where we have inserted the z dependence used in (17). Then using the spectral flow (91), we obtain the following graviton orbit

$$NS_0 = A_1^2 (\theta_{0,6} + \theta_{6,6}) + C_1^2 (\theta_{2,6} + \theta_{8,6}) + B_1^2 (\theta_{4,6} + \theta_{10,6}), \quad (35)$$

where again we omit the factor of η for simplicity.

Furthermore we can close the orbit of this theory under S modular transformation using the additional spectral flow invariant orbit

$$NS_1 = B_1 C_1 (\theta_{0,6} + \theta_{6,6}) + A_1 B_1 (\theta_{2,6} + \theta_{8,6}) + C_1 A_1 (\theta_{4,6} + \theta_{10,6}). \quad (36)$$

Then the S modular transformation is summarized by the following S matrix

$$S_{ij} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}, \quad i, j = 0, 1, \quad (37)$$

which acts on NS_0, NS_1 as in (23).

Now we can easily construct the modular invariant partition function as reviewed in section 3. Let us define the supersymmetric characters

$$X_i(\tau, z) = \left(\theta_3^3 NS_i - \theta_4^3 \widetilde{NS}_i - \theta_2^3 R_i - i\theta_1^3 \widetilde{R}_i \right) (\tau, z), \quad i = 0, 1, \quad (38)$$

where $\widetilde{NS}_i, R_i, \widetilde{R}_i$ are obtained by the spectral flow (91). Using these supersymmetric characters, we can write down the modular invariant partition function

$$Z_{GSO}^{A_2 \otimes A_2}(\tau, \bar{\tau}) = \frac{1}{|\eta(\tau)|^8} \left(|X_0|^2 + 2|X_1|^2 \right) (\tau, \bar{\tau}). \quad (39)$$

How can we see the structure of D_4 ? In fact, note that the S matrix (37) is equivalent to that for the $F_0 + F_4, F_2$ pieces of D_4 theory in (28). Thus we claim that the following equations should hold

$$NS_0 = F_0^{(NS)} + F_4^{(NS)}, \quad NS_1 = F_2^{(NS)}, \quad (40)$$

where (NS) denotes the piece of NS sector in F_ℓ (16). We have checked that the explicit q -expansion have the same form in both sides. Moreover one can check the equivalence in the other sector. Thus we can say that we have reproduced the block diagonal elements of D_4 modular invariant in terms of the spectral flow invariant orbits of $A_2 \otimes A_2$ Gepner model, and observed the equivalence : $Z_{GSO}^{D_4} = Z_{GSO}^{A_2 \otimes A_2}$.

In fact, the level 6 theta functions $\theta_{m,6}(\tau, 0)$ are functionally independent for different $|m|$. Thus we suspect that there must be the following equivalence relation between the coefficients of each theta function

$$\begin{aligned} \text{Ch}_{0,0}^{NS,(4)} + \text{Ch}_{4,0}^{NS,(4)} &= A_1^2, & \text{Ch}_{2,0}^{NS,(4)} &= B_1 C_1, \\ \text{Ch}_{4,-4}^{NS,(4)} + \text{Ch}_{4,2}^{NS,(4)} &= C_1^2, & \text{Ch}_{2,2}^{NS,(4)} &= A_1 B_1, \\ \text{Ch}_{4,-2}^{NS,(4)} + \text{Ch}_{4,4}^{NS,(4)} &= B_1^2, & \text{Ch}_{2,-2}^{NS,(4)} &= C_1 A_1. \end{aligned} \quad (41)$$

We can give a complete proof for these identities between the characters of minimal models at different levels. We show this in Appendix B. The identity for other sectors can be obtained in a trivial way.

For the singular Calabi-Yau 3,4-folds, the building block in [11] remains invariant under the spectral flow. These spectral flow invariant orbits are combined to make modular invariant partition function using modular invariant of $\widehat{SU}(2)$ and theta system. The additional modular invariant for the theta system is irrelevant to the relation between D_4 and $A_2 \otimes A_2$. Thus the extension to the singular 3,4-fold is straightforward due to the identity for minimal models (41).

4.2 E_6 case

Let us consider the E_6 modular invariant in (19)

$$Z_{GSO}^{E_6} = \frac{1}{|\eta|^8} \left(|F_0 + F_6|^2 + |F_4 + F_{10}|^2 + |F_3 + F_7|^2 \right). \quad (42)$$

We wish to make the similar modular invariants using the $A_2 \otimes A_3$ Gepner model.

We label the characters of six irreducible representations in the NS sector for A_3 minimal

model at level two in the following way

$$A_2 = \text{Ch}_{0,0}^{NS,(2)}, \quad B_2 = \text{Ch}_{2,2}^{NS,(2)}, \quad C_2 = \text{Ch}_{2,0}^{NS,(2)}, \quad D_2 = \text{Ch}_{2,-2}^{NS,(2)}, \quad E_2 = \text{Ch}_{1,1}^{NS,(2)}, \quad F_2 = \text{Ch}_{1,-1}^{NS,(2)}. \quad (43)$$

Now we use A_2 for both the label of level one minimal model and that of the character in equation (43), but there may be no confusion. Under the spectral flow with $\eta = 1$ in (91), the above characters change as follows

$$A_2 \rightarrow B_2 \rightarrow C_2 \rightarrow D_2 \rightarrow A_2, \quad (44)$$

$$E_2 \rightarrow F_2 \rightarrow E_2. \quad (45)$$

Next we specify the condition of GSO projection in NS sector as follows

$$F + F_{MM_1} + F_{MM_2} + \frac{m_1}{3} + \frac{m_2}{4} - \frac{p}{\sqrt{6}} \in 2\mathbf{Z} + 1, \quad (46)$$

where MM_1, MM_2 represent A_2, A_3 minimal models, respectively, and R sector has the obvious condition. Then let us calculate the trace over the left-moving Hilbert space. Consider the NS sector with $F + F_{MM_1} + F_{MM_2} \in 2\mathbf{Z} + 1$. The sum over the momenta becomes

$$\sum_n q^{\frac{1}{2}p^2} = \sum_n q^{\frac{6}{2}\left(2n + \frac{4m_1+3m_2}{12}\right)^2} = \sum_n q^{12\left(n + \frac{4m_1+3m_2}{24}\right)^2} = \theta_{4m_1+3m_2,12}(\tau). \quad (47)$$

On the other hand, in the NS sector with $F + F_{MM_1} + F_{MM_2} \in 2\mathbf{Z}$, we obtain the following sum

$$\sum_n q^{\frac{1}{2}p^2} = \sum_n q^{\frac{6}{2}\left(2n+1 + \frac{4m_1+3m_2}{12}\right)^2} = \sum_n q^{12\left(n + \frac{4m_1+3m_2+12}{24}\right)^2} = \theta_{4m_1+3m_2+12,12}(\tau). \quad (48)$$

We have to make the spectral flow invariant orbit and modular invariant of the tensor product theory as D_4 case. Again we adopt the simplest ansatz that the graviton orbit includes the term

$$A_1 A_2 (\tau, z) \theta_{0,12}(\tau, -z/6), \quad (49)$$

where the expected z -dependence has been included. Then using the spectral flow (91), we obtain the following graviton orbit

$$\begin{aligned} \text{NS}_0 = & A_1 A_2 \theta_{0,12} + C_1 D_2 \theta_{2,12} + B_1 C_2 \theta_{4,12} + A_1 B_2 \theta_{6,12} \\ & + C_1 A_2 \theta_{8,12} + B_1 D_2 \theta_{10,12} + A_1 C_2 \theta_{12,12} + C_1 B_2 \theta_{14,12} \\ & + B_1 A_2 \theta_{16,12} + A_1 D_2 \theta_{18,12} + C_1 C_2 \theta_{20,12} + B_1 B_2 \theta_{22,12}. \end{aligned} \quad (50)$$

Then we find that we can close the S modular transformation using additional spectral flow invariant orbits

$$\text{NS}_1 = A_1 C_2 \theta_{0,12} + C_1 B_2 \theta_{2,12} + B_1 A_2 \theta_{4,12} + A_1 D_2 \theta_{6,12}$$

$$\begin{aligned}
& +C_1C_2\theta_{8,12} + B_1B_2\theta_{10,12} + A_1A_2\theta_{12,12} + C_1D_2\theta_{14,12} \\
& +B_1C_2\theta_{16,12} + A_1B_2\theta_{18,12} + C_1A_2\theta_{20,12} + B_1D_2\theta_{22,12},
\end{aligned} \tag{51}$$

$$\begin{aligned}
\text{NS}_2 = & B_1F_2(\theta_{1,12} + \theta_{13,12}) + A_1E_2(\theta_{3,12} + \theta_{15,12}) \\
& +C_1F_2(\theta_{5,12} + \theta_{17,12}) + B_1E_2(\theta_{7,12} + \theta_{19,12}) \\
& +A_1F_2(\theta_{9,12} + \theta_{21,12}) + C_1E_2(\theta_{11,12} + \theta_{23,12}).
\end{aligned} \tag{52}$$

Then S transformation is summarized as the following S matrices,

$$S_{ij} = \frac{1}{2} \begin{pmatrix} 1 & 1 & \sqrt{2} \\ 1 & 1 & -\sqrt{2} \\ \sqrt{2} & -\sqrt{2} & 0 \end{pmatrix}, \quad i, j = 0, 1, 2. \tag{53}$$

Then we can obtain the modular invariant partition function using the supersymmetric characters

$$Z_{GSO}^{A_2 \otimes A_3}(\tau, \bar{\tau}) = \frac{1}{|\eta(\tau)|^8} (|X_0|^2 + |X_1|^2 + |X_2|^2) (\tau, \bar{\tau}). \tag{54}$$

Note that the S matrix (53) is equivalent to that for the block diagonal pieces $F_0 + F_6, F_4 + F_{10}, F_3 + F_7$ of E_6 modular invariant theory (42). Thus we claim that the following equations should hold

$$NS_0 = F_0^{(NS)} + F_6^{(NS)}, \quad NS_1 = F_4^{(NS)} + F_{10}^{(NS)}, \quad NS_2 = F_3^{(NS)} + F_7^{(NS)}, \tag{55}$$

where (NS) denotes the contribution of NS sector in (16). Again we have checked that the explicit q expansion have the same form. Also we can check the equivalence in the other sector via the explicit q -expansion and modular property. Thus we have reproduced the block diagonal elements of E_6 modular invariants in terms of the spectral flow invariant orbits by $A_1 \otimes A_2$ tensor products.

Note that the pattern of multiplication of theta function in each spectral flow invariant orbit is different between NS_0, NS_1 and NS_2 . The number of element is different in two flow invariant from minimal models, using (30), (44) or (30), (45). Also $F_0 + F_6, F_4 + F_{10}$ in (42) do not close by itself under field identification in minimal model, but $F_3 + F_7$ closes by itself.

In fact, the level-12 theta functions $\theta_{m,12}(\tau, 0)$ are functionally independent for different $|m|$, thus we can expect that there should be the equivalence relation between the characters in minimal models, such as

$$\text{Ch}_{0,0}^{NS,(10)} + \text{Ch}_{6,0}^{NS,(10)} = A_1A_2. \tag{56}$$

The other equations like this are easily obtained. Furthermore we can prove the identity exactly (Appendix B).

4.3 E_8 case

We can proceed in the same way as the D_4, E_6 case. In this case, we wish to construct modular invariants of $A_2 \otimes A_4$ tensor product, and reproduce the structure of E_8 modular invariant theory in (19)

$$Z_{GSO}^{E_8} = \frac{1}{|\eta|^8} \left(|F_0 + F_{10} + F_{18} + F_{28}|^2 + |F_6 + F_{12} + F_{16} + F_{22}|^2 \right). \quad (57)$$

First we label the NS characters of ten irreducible representations of A_4 minimal model at level 3 as follows

$$\begin{aligned} A_3 &= \text{Ch}_{0,0}^{NS,(3)}, \quad B_3 = \text{Ch}_{3,3}^{NS,(3)}, \quad C_3 = \text{Ch}_{3,1}^{NS,(3)}, \quad D_3 = \text{Ch}_{3,-1}^{NS,(3)}, \quad E_3 = \text{Ch}_{3,-3}^{NS,(3)}, \\ F_3 &= \text{Ch}_{1,1}^{NS,(3)}, \quad G_3 = \text{Ch}_{1,-1}^{NS,(3)}, \quad H_3 = \text{Ch}_{2,2}^{NS,(3)}, \quad I_3 = \text{Ch}_{2,0}^{NS,(3)}, \quad J_3 = \text{Ch}_{2,-2}^{NS,(3)}. \end{aligned} \quad (58)$$

Then we can summarize the action of spectral flow in the following manner

$$A_3 \rightarrow B_3 \rightarrow C_3 \rightarrow D_3 \rightarrow E_3 \rightarrow A_3, \quad (59)$$

$$F_3 \rightarrow G_3 \rightarrow H_3 \rightarrow I_3 \rightarrow J_3 \rightarrow F_3, \quad (60)$$

and there exist two naive spectral flow invariant orbits using (30), (59), (60) in $A_2 \otimes A_4$ theory.

The GSO projection in the NS sector is given by

$$F + F_{MM_1} + F_{MM_2} + \frac{m_1}{3} + \frac{m_3}{5} - \frac{p}{\sqrt{15}} \in 2\mathbf{Z} + 1, \quad (61)$$

where we denote A_2, A_4 minimal models by MM_1, MM_2 respectively, and R sector has the similar condition. Again, consider the NS sector with $F + F_{MM_1} + F_{MM_2} \in 2\mathbf{Z} + 1$. The sum over the momenta becomes

$$\sum q^{\frac{1}{2}p^2} = \sum_n q^{\frac{15}{2} \left(2n + \frac{5m_1+3m_3}{30} \right)^2} = \sum_n q^{30 \left(n + \frac{5m_1+3m_3}{60} \right)^2} = \theta_{5m_1+3m_3,30}(\tau). \quad (62)$$

For the NS setor with $F + F_{MM_1} + F_{MM_2} \in 2\mathbf{Z}$, we obtain the following sum

$$\sum q^{\frac{1}{2}p^2} = \sum_n q^{\frac{30}{4} \left(2n+1 + \frac{5m_1+3m_3}{30} \right)^2} = \sum_n q^{30 \left(n + \frac{5m_1+3m_3+30}{60} \right)^2} = \theta_{5m_1+3m_3+30,30}(\tau). \quad (63)$$

Then the same way as previous subsection, we can make the following graviton orbit

$$\begin{aligned}
NS_0 = & A_1 A_3 (\theta_{0,30} + \theta_{30,30}) + C_1 E_3 (\theta_{2,30} + \theta_{32,30}) + B_1 D_3 (\theta_{4,30} + \theta_{34,30}) \\
& + A_1 C_3 (\theta_{6,30} + \theta_{36,30}) + C_1 B_3 (\theta_{8,30} + \theta_{38,30}) + B_1 A_3 (\theta_{10,30} + \theta_{40,30}) \\
& + A_1 E_3 (\theta_{12,30} + \theta_{42,30}) + C_1 D_3 (\theta_{14,30} + \theta_{44,30}) + B_1 C_3 (\theta_{16,30} + \theta_{46,30}) \\
& + A_1 B_3 (\theta_{18,30} + \theta_{48,30}) + C_1 A_3 (\theta_{20,30} + \theta_{50,30}) + B_1 E_3 (\theta_{22,30} + \theta_{52,30}) \\
& + A_1 D_3 (\theta_{24,30} + \theta_{54,30}) + C_1 C_3 (\theta_{26,30} + \theta_{56,30}) + B_1 B_3 (\theta_{28,30} + \theta_{58,30}). \quad (64)
\end{aligned}$$

Then we can close the system under S transformation using the additional spectral flow orbit

$$\begin{aligned}
NS_1 = & A_1 I_3 (\theta_{0,30} + \theta_{30,30}) + C_1 H_3 (\theta_{2,30} + \theta_{32,30}) + B_1 G_3 (\theta_{4,30} + \theta_{34,30}) \\
& + A_1 F_3 (\theta_{6,30} + \theta_{36,30}) + C_1 J_3 (\theta_{8,30} + \theta_{38,30}) + B_1 I_3 (\theta_{10,30} + \theta_{40,30}) \\
& + A_1 H_3 (\theta_{12,30} + \theta_{42,30}) + C_1 G_3 (\theta_{14,30} + \theta_{44,30}) + B_1 F_3 (\theta_{16,30} + \theta_{46,30}) \\
& + A_1 J_3 (\theta_{18,30} + \theta_{48,30}) + C_1 I_3 (\theta_{20,30} + \theta_{50,30}) + B_1 H_3 (\theta_{22,30} + \theta_{52,30}) \\
& + A_1 G_3 (\theta_{24,30} + \theta_{54,30}) + C_1 F_3 (\theta_{26,30} + \theta_{56,30}) + B_1 J_3 (\theta_{28,30} + \theta_{58,30}). \quad (65)
\end{aligned}$$

Notice that NS_0, NS_1 uses the spectral flow invariants (30), (59) and (30), (60), respectively. The S matrix for these orbits coincides with that for the block diagonal term $F_0 + F_{10} + F_{18} + F_{28}, F_6 + F_{12} + F_{18} + F_{22}$ in E_8 invariants (57)

$$S_{ij} = \frac{2}{\sqrt{5}} \begin{pmatrix} \frac{\sqrt{10-2\sqrt{5}}}{4} & \frac{\sqrt{10+2\sqrt{5}}}{4} \\ \frac{\sqrt{10+2\sqrt{5}}}{4} & -\frac{\sqrt{10-2\sqrt{5}}}{4} \end{pmatrix}, \quad i, j = 0, 1. \quad (66)$$

Thus we can write down modular invariant partition function using supersymmetric characters

$$Z_{GSO}^{A_2 \otimes A_4}(\tau, \bar{\tau}) = \frac{1}{|\eta(\tau)|^8} (|X_0|^2 + |X_1|^2) (\tau, \bar{\tau}). \quad (67)$$

In the same way as previous cases, we can write down the following relation.

$$NS_0 = F_0^{(NS)} + F_{10}^{(NS)} + F_{18}^{(NS)} + F_{28}^{(NS)}, \quad (68)$$

$$NS_1 = F_6^{(NS)} + F_{12}^{(NS)} + F_{16}^{(NS)} + F_{22}^{(NS)}, \quad (69)$$

where (NS) denotes the contribution from NS sector in (16). We have compared explicit q expansions in both side and checked the equivalence. Thus in the same sense as the previous subsections, we have succeeded to rewrite the block diagonal elements in E_8 invariants in terms of the spectral flow invariant orbits of $A_2 \otimes A_4$ theory. Compared with E_6 case where two naive spectral flow orbits in minimal model divided into three orbits in whole theory, we have only

two orbits in whole system related to each orbits in minimal models. This is the same manner as in D_4 case.

Moreover using the fact that the level-30 theta functions $\theta_{m,30}(\tau, 0)$ are functionally independent for different $|m|$, we can read off the identities among the characters in minimal models contained in (68), (69), such as

$$\text{Ch}_{0,0}^{NS,(28)} + \text{Ch}_{10,0}^{NS,(28)} + \text{Ch}_{18,0}^{NS,(28)} + \text{Ch}_{28,0}^{NS,(28)} = A_1 A_3. \quad (70)$$

All the relation like this are easily obtained. Again we can give a exact proof of the identity, see Appendix B.

5 Conclusion and discussion

In this paper, we have studied the toroidal partition functions of non-critical superstring theory on $\mathbf{R}^{d-1,1} \times (\mathbf{R}_\phi \times \mathbf{S}^1) \times M_{D_4, E_6, E_8}$, which is conjectured to give the dual description of Calabi-Yau manifolds with the ADE singularity in the decoupling limit. The ADE classification of modular invariants associated to the type of Calabi-Yau singularities suggests that the natural reinterpretation of D_4, E_6, E_8 theory via Gepner models of $A_2 \otimes A_2, A_2 \otimes A_3, A_2 \otimes A_4$. Strategy of the spectral flow invariant orbit has given more natural framework to the singular Calabi-Yau compactification than the conventional smooth Calabi-Yau compactification. Moreover we have obtained the identities among the characters in the minimal models at different levels. Maybe the existence of these identities were implicitly known in the work [14]. But in the present more realistic situation than only the minimal models, we have been able to obtain the relations more naturally. Furthermore we have given the complete proof of the identities. Our work gives the basic consistency checks on the use of Landau-Ginzburg theory for the singular Calabi-Yau compactification.

The characters of minimal models are defined by taking care of all the null states. Thus our identity among the characters of minimal models at different levels may seem to be rather non-trivial. However, in CFTs, we often encounter the phenomena that we can obtain the non-trivial relation in some model by imposing the larger symmetry. There would be some interest to investigate the precise structure of the identity between each representations along the null field construction [30].

Here we pose the unresolved problem. We can calculate the elliptic genus of the singular Calabi-Yau manifold using the CFTs. It turns out that the elliptic genus vanishes [13]. This

fact is reflected in the following observation. For example, in the case of K3 surface with the isolated ADE singularity, we cannot reproduce any nontrivial Hodge number [31] of the corresponding ALE space. Thus the CFT system really does not respect the geometry of the ALE spaces. Only the exception is the case of conifold [12] where extra Hodge number has been appeared, it was claimed that it correspond to an additional massless soliton as in [32]. On the other hand in the smooth CFTs, D-brane wrapped around a collapsing cycle becomes a fractional brane [33] with a finite mass. Thus perturbative description is reliable at least if the string coupling is small and the mass of fractional brane is large. Then it would be unreasonable to claim that in the singular CFTs the partition function includes the extra massless mode. A sensible interpretation on the whole phenomena is not clear.

Moreover it would be interesting to consider the boundary states in these backgrounds like [34, 35], and investigate the relation to Seiberg-Witten theory as in [36]. Then, only the nontrivial part would be the construction and interpretation of boundary states for the Liouville sector [37].

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Appendix A Convention of Conformal Field Theory

In this appendix, we summarize the notation and collect the formulas used in this paper. We set $q = e^{2\pi i\tau}$ and $y = e^{2\pi iz}$.

1. Theta functions

Jacobi theta functions are defined by

$$\begin{aligned}\theta_1(\tau, z) &= i \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{1}{2}(n-\frac{1}{2})^2} y^{n-\frac{1}{2}} = 2 q^{\frac{1}{8}} \sin(\pi z) \prod_{m=1}^{\infty} (1 - q^m)(1 - yq^m)(1 - y^{-1}q^m), \\ \theta_2(\tau, z) &= \sum_{n=-\infty}^{\infty} q^{\frac{1}{2}(n-\frac{1}{2})^2} y^{n-\frac{1}{2}} = 2 q^{\frac{1}{8}} \cos(\pi z) \prod_{m=1}^{\infty} (1 - q^m)(1 + yq^m)(1 + y^{-1}q^m), \\ \theta_3(\tau, z) &= \sum_{n=-\infty}^{\infty} q^{\frac{n^2}{2}} y^n = \prod_{m=1}^{\infty} (1 - q^m)(1 + yq^{m-\frac{1}{2}})(1 + y^{-1}q^{m-\frac{1}{2}}), \\ \theta_4(\tau, z) &= \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{n^2}{2}} y^n = \prod_{m=1}^{\infty} (1 - q^m)(1 - yq^{m-\frac{1}{2}})(1 - y^{-1}q^{m-\frac{1}{2}}).\end{aligned}\tag{71}$$

For a positive integer k , theta function of level k is defined by

$$\theta_{m,k}(\tau, z) = \sum_{n=-\infty}^{\infty} q^{k(n+\frac{m}{2k})^2} y^{k(n+\frac{m}{2k})},\tag{72}$$

where $m \in \mathbf{Z}_{2k}$. We can rewrite the Jacobi theta functions in terms of the theta function of level 2

$$\begin{aligned}i\theta_1 &= \theta_{1,2} - \theta_{3,2}, & \theta_2 &= \theta_{1,2} + \theta_{3,2}, \\ \theta_3 &= \theta_{0,2} + \theta_{2,2}, & \theta_4 &= \theta_{0,2} - \theta_{2,2}.\end{aligned}\tag{73}$$

Dedekind η function is represented as

$$\eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n).\tag{74}$$

2. Characters of $N = 2$ minimal model

There is the discrete series of unitary representations of $N = 2$ superconformal algebra with $c < 3$, in fact with $c = \frac{3k}{k+2}$ ($k = N - 2 = 1, 2, 3, \dots$). Based on these representations, one can construct families of conformal field theories known as $N = 2$ minimal models. Their highest

weight states are characterized by conformal weight h and the $U(1)$ charge q :

$$h_m^{\ell,s} = \frac{\ell(\ell+2) - m^2}{4(k+2)} + \frac{s^2}{8}, \quad q_m^{\ell,s} = \frac{m}{k+2} - \frac{s}{2}, \quad (75)$$

where $\ell \in \{0, \dots, k\}$, $|m-s| \leq \ell$, $s \in \{-1, 0, 1, 2\}$ and $\ell + m + s \equiv 0 \pmod{2}$. This range of (ℓ, m, s) is called ‘standard range’.

The discrete representations of $N = 2$ algebra are related to the $\widehat{SU(2)}_k$ representations. The character of $\widehat{SU(2)}_k$ with the spin $\frac{\ell}{2}$ ($0 \leq \ell \leq k$) representation is defined by

$$\chi_\ell^{(k)}(\tau, z) = \frac{\theta_{\ell+1,k+2} - \theta_{-\ell-1,k+2}}{\theta_{1,2} - \theta_{-1,2}}(\tau, z) := \sum_{m \in \mathbf{Z}_{2k}} c_m^\ell(\tau) \theta_{m,k}(\tau, z), \quad (76)$$

where we refer to the coefficient $c_m^\ell(\tau)$ as string function. String function has the following properties : $c_m^\ell = c_{-m}^\ell = c_{m+2k}^\ell = c_{m+k}^{k-\ell}$ and $c_m^\ell = 0$ unless $\ell + m \equiv 0 \pmod{2}$.

On the other side, the character of $N = 2$ representation labeled by (ℓ, m, s) is defined by $\chi_{m,s}^\ell(\tau, z) = \text{Tr}_{\mathcal{H}_{m,s}^\ell} q^{L_0 - \frac{c}{24}} y^{J_0}$. The explicit formula of $N = 2$ character is obtained through the branching relation [3, 28]

$$\chi_\ell^{(k)}(\tau, w) \theta_{s,2}(\tau, w - z) = \sum_{m=-k-1}^{k+2} \chi_m^{\ell,s}(\tau, z) \theta_{m,k+2}\left(\tau, w - \frac{2z}{k+2}\right), \quad (77)$$

and is given by [38]

$$\chi_m^{\ell,s}(\tau, z) = \sum_{r \in \mathbf{Z}_k} c_{m-s+4r}^\ell(\tau) \theta_{2m+(k+2)(-s+4r), 2k(k+2)}\left(\tau, \frac{z}{k+2}\right). \quad (78)$$

This character is actually defined in the ‘extended range’

$$\ell \in \{0, \dots, k\}, \quad m \in \mathbf{Z}_{2k+4}, \quad s \in \mathbf{Z}_4 \quad \text{and} \quad \ell + m + s \equiv 0 \pmod{2}. \quad (79)$$

However, since the character has the following properties

$$\chi_m^{\ell,s} = \chi_{m+2k+4}^{\ell,s} = \chi_m^{\ell,s+4} = \chi_{m+k+2}^{k-\ell,s+2} \quad \text{and} \quad \chi_m^{\ell,s} = 0 \quad \text{unless} \quad \ell + m + s \equiv 0 \pmod{2}, \quad (80)$$

we can always bring the range of (ℓ, m, s) into the standard range.

The characters of $N = 2$ minimal model of level k are defined by

$$\begin{aligned} \text{Ch}_{\ell,m}^{NS,(k)}(\tau, z) &= \chi_m^{\ell,0}(\tau, z) + \chi_m^{\ell,2}(\tau, z), & \widetilde{\text{Ch}}_{\ell,m}^{NS,(k)}(\tau, z) &= \chi_m^{\ell,0}(\tau, z) - \chi_m^{\ell,2}(\tau, z), \\ \text{Ch}_{\ell,m}^{R,(k)}(\tau, z) &= \chi_m^{\ell,1}(\tau, z) + \chi_m^{\ell,3}(\tau, z), & \widetilde{\text{Ch}}_{\ell,m}^{R,(k)}(\tau, z) &= \chi_m^{\ell,1}(\tau, z) - \chi_m^{\ell,3}(\tau, z). \end{aligned} \quad (81)$$

The explicit formula of the character in NS sector is represented as an infinite product form [39]

$$\text{Ch}_{\ell,m}^{NS,(k)}(\tau, z) = q^{h_{\ell,m}^{(NS)} - \frac{c}{24}} y^{q_{\ell,m}^{(NS)}} \prod_{n=1}^{\infty} \frac{(1 + yq^{n-1/2})(1 + y^{-1}q^{n-1/2})}{(1 - q^n)^2} \Gamma_{\ell,m}^{(N)}, \quad (82)$$

where $N = k + 2$ and $h_{\ell,m}^{(NS)} = h_m^{\ell,0}$, $q_{\ell,m}^{(NS)} = q_m^{\ell,0}$,

$$\Gamma_{\ell,m}^{(N)} = \prod_{n=1}^{\infty} \frac{(1 - q^{Nn+\ell+1-N})(1 - q^{Nn-\ell-1})(1 - q^{Nn})^2}{(1 + yq^{Nn-\frac{\ell+m+1}{2}})(1 + y^{-1}q^{Nn+\frac{\ell+m+1}{2}-N})(1 + y^{-1}q^{Nn-\frac{\ell-m+1}{2}})(1 + yq^{Nn+\frac{\ell-m+1}{2}-N})}.$$

3. Modular transformations

For simplicity, we use the following abbreviations : $\theta_{m,k}(\tau) \equiv \theta_{m,k}(\tau, 0)$, $\chi_m^{\ell,s}(\tau) \equiv \chi_m^{\ell,s}(\tau, 0)$. Under the modular transformation $S : \tau \rightarrow -1/\tau$, the characters defined above transform as

$$\chi_{\ell}^{(k)}(-1/\tau) = \sum_{\ell'=0}^k S_{\ell\ell'}^{(k)} \chi_{\ell'}^{(k)}(\tau), \quad (83)$$

$$\theta_{m,k}(-1/\tau) = \sqrt{-i\tau} \sum_{m' \in \mathbf{Z}_{2k}} \tilde{S}_{mm'}^{(k)} \theta_{m',k}(\tau), \quad (84)$$

$$\chi_m^{\ell,s}(-1/\tau) = \sum_{\ell',m',s'} S_{\ell\ell'}^{(k)} \tilde{S}_{mm'}^{(k+2)\dagger} \tilde{S}_{ss'}^{(2)} \chi_{m'}^{\ell',s'}(\tau), \quad (85)$$

where $\sum_{\ell',m',s'}$ denotes the summation over the extended range (79). The modular transformation matrices of the characters are given by

$$S_{\ell\ell'}^{(k)} = \sqrt{\frac{2}{k+2}} \sin \pi \frac{(\ell+1)(\ell'+1)}{k+2}, \quad (86)$$

$$\tilde{S}_{mm'}^{(k)} = \frac{1}{\sqrt{2k}} e^{-2\pi i \frac{mm'}{2k}}. \quad (87)$$

Under the modular transformation $T : \tau \rightarrow \tau + 1$, the characters transform as

$$\chi_{\ell}^{(k)}(\tau + 1) = e^{2\pi i \left[\frac{\ell(\ell+2)}{4(k+2)} - \frac{c}{24} \right]} \chi_{\ell}^{(k)}(\tau), \quad (88)$$

$$\theta_{m,k}(\tau + 1) = e^{2\pi i \frac{m^2}{4k}} \theta_{m,k}(\tau), \quad (89)$$

$$\chi_m^{\ell,s}(\tau + 1) = e^{2\pi i \left[h_m^{\ell,s} - \frac{c}{24} \right]} \chi_m^{\ell,s}(\tau), \quad (90)$$

where $c = \frac{3k}{k+2}$.

If we want to know how the characters transform under the spectral flow, we have only to know the properties of the characters under the shift of parameter z . Then the characters $\chi_m^{\ell,s}(\tau, z)$ and $\theta_{m,N}(\tau, -2z/N)$ transform as

$$\chi_m^{\ell,s} \left(\tau, z + \frac{\tau}{2} \right) = q^{-\frac{c}{24}} y^{-\frac{c}{6}} \chi_{m-1}^{\ell,s-1}(\tau, z),$$

$$\begin{aligned}
\chi_m^{\ell,s}(\tau, z + \tau) &= q^{-\frac{c}{6}} y^{-\frac{c}{3}} \chi_{m-2}^{\ell,s}(\tau, z), \\
\chi_m^{\ell,s}\left(\tau, z + \frac{1}{2}\right) &= (-i)^s e^{\frac{i\pi m}{N}} \chi_m^{\ell,s}(\tau, z), \\
\theta_{m,N}\left(\tau, -\frac{2(z+\tau)}{N}\right) &= q^{-\frac{1}{N}} y^{-\frac{2}{N}} \theta_{m-2,N}\left(\tau, -\frac{2z}{N}\right), \\
\theta_{m,N}\left(\tau, -\frac{2(z+\tau/2)}{N}\right) &= q^{-\frac{1}{4N}} y^{-\frac{1}{N}} \theta_{m-1,N}\left(\tau, -\frac{2z}{N}\right), \\
\theta_{m,N}\left(\tau, -\frac{2(z+1/2)}{N}\right) &= e^{-\frac{i\pi m}{N}} \theta_{m-1,N}\left(\tau, -\frac{2z}{N}\right),
\end{aligned} \tag{91}$$

where $N = k + 2$.

Appendix B Proof of identities between minimal models

In this appendix, we give exact proofs of the identity (41), (56), (70) among characters of minimal models. The other identity can be proven along the same line.

1. D_4 case

Let us consider the first identity in (41). We use the infinite product representation of minimal characters at level 4, 1 (82)

$$\begin{aligned}
\text{Ch}_{0,0}^{NS,(4)}(\tau) &= q^{-\frac{1}{12}} \prod_{n=1}^{\infty} \frac{(1 + q^{n-\frac{1}{2}})^2}{(1 - q^n)^2} \cdot \frac{(1 - q^{6n-5})(1 - q^{6n-1})(1 - q^{6n})^2}{(1 + q^{6n-\frac{1}{2}})^2 (1 + q^{6n-\frac{11}{2}})^2}, \\
\text{Ch}_{4,0}^{NS,(4)}(\tau) &= q^{\frac{11}{12}} \prod_{n=1}^{\infty} \frac{(1 + q^{n-\frac{1}{2}})^2}{(1 - q^n)^2} \cdot \frac{(1 - q^{6n-5})(1 - q^{6n-1})(1 - q^{6n})^2}{(1 + q^{6n-\frac{5}{2}})^2 (1 + q^{6n-\frac{7}{2}})^2}, \\
\text{Ch}_{0,0}^{NS,(1)}(\tau) &= q^{-\frac{1}{24}} \prod_{n=1}^{\infty} \frac{(1 + q^{n-\frac{1}{2}})^2}{(1 - q^n)^2} \cdot \frac{(1 - q^{3n-2})(1 - q^{3n-1})(1 - q^{3n})^2}{(1 + q^{3n-\frac{1}{2}})^2 (1 + q^{3n-\frac{5}{2}})^2},
\end{aligned} \tag{92}$$

where we set $z = 0$ for simplicity. Dividing by $\prod_{n=1}^{\infty} (1 + q^{n-\frac{1}{2}})^2 / (1 - q^n)^2$, we can write down the identity (41): $\text{Ch}_{0,0}^{NS,(4)} \text{Ch}_{4,0}^{NS,(4)} = \text{Ch}_{0,0}^{NS,(1)}$ as follows

$$\begin{aligned}
&\prod_{n=1}^{\infty} (1 - q^{6n-5})(1 - q^{6n-1})(1 - q^{6n})^2 \\
&\times \left[\prod_{n=1}^{\infty} \frac{1}{(1 + q^{6n-\frac{1}{2}})^2 (1 + q^{6n-\frac{11}{2}})^2} + q \prod_{n=1}^{\infty} \frac{1}{(1 + q^{6n-\frac{5}{2}})^2 (1 + q^{6n-\frac{7}{2}})^2} \right]
\end{aligned} \tag{93}$$

$$= \prod_{n=1}^{\infty} \frac{(1+q^{n-\frac{1}{2}})^2}{(1-q^n)^2} \cdot \frac{(1-q^{3n-2})^2 (1-q^{3n-1})^2 (1-q^{3n})^4}{(1+q^{3n-\frac{1}{2}})^4 (1+q^{3n-\frac{5}{2}})^4}.$$

We can rewrite the right hand side into the following form

$$\prod_{n=1}^{\infty} \frac{(1-q^{6n})^2 (1-q^{6n-3})^2 (1+q^{6n-\frac{3}{2}})^2 (1+q^{6n-\frac{9}{2}})^2}{(1+q^{6n-\frac{1}{2}})^2 (1+q^{6n-\frac{5}{2}})^2 (1+q^{6n-\frac{7}{2}})^2 (1+q^{6n-\frac{11}{2}})^2}. \quad (94)$$

Then dividing the both sides in (93) by $\prod_{n=1}^{\infty} (1-q^{6n})^2$, the first identity in (41) can be rewritten as

$$\begin{aligned} & \prod_{n=1}^{\infty} (1-q^{6n-1})(1-q^{6n-5})(1+q^{6n-\frac{5}{2}})^2(1+q^{6n-\frac{7}{2}})^2 \\ & + q \prod_{n=1}^{\infty} (1-q^{6n-1})(1-q^{6n-5})(1+q^{6n-\frac{1}{2}})^2(1+q^{6n-\frac{11}{2}})^2 \\ & = \prod_{n=1}^{\infty} (1-q^{6n-3})^2(1+q^{6n-\frac{3}{2}})^2(1+q^{6n-\frac{9}{2}})^2, \end{aligned} \quad (95)$$

by cancelling the terms in the denominator. Using the Jacobi triple product identity

$$\sum_{n=-\infty}^{\infty} q^{n^2} y^n = \prod_{m=1}^{\infty} (1-q^{2m})(1+yq^{2m-1})(1+y^{-1}q^{2m-1}), \quad (96)$$

with $q \rightarrow q^3$ and $y = -q^2, q^{\frac{1}{2}}, q^{\frac{5}{2}}, -1, q^{\frac{3}{2}}$, we can rewrite (95) as

$$\begin{aligned} & \left(\sum_{n=-\infty}^{\infty} (-1)^n q^{3n^2+2n} \right) \left[\left(\sum_{n=-\infty}^{\infty} q^{3n^2+\frac{1}{2}n} \right)^2 + q \left(\sum_{n=-\infty}^{\infty} q^{3n^2+\frac{5}{2}n} \right)^2 \right] \\ & = \left(\sum_{n=-\infty}^{\infty} (-1)^n q^{3n^2} \right) \left(\sum_{n=-\infty}^{\infty} q^{3n^2+\frac{3}{2}n} \right)^2. \end{aligned} \quad (97)$$

Then we can drop the unwanted factor q in the second term in the left hand side. Using

$$\begin{aligned} \sum_{n=-\infty}^{\infty} q^{k(n+\frac{m}{4k})^2} &= \theta_{m,4k}(\tau) + \theta_{m+4k,4k}(\tau), \\ \sum_{n=-\infty}^{\infty} (-1)^n q^{k(n+\frac{m}{4k})^2} &= \theta_{m,4k}(\tau) - \theta_{m+4k,4k}(\tau), \end{aligned} \quad (98)$$

and $\theta_{m,k}(\tau) = \theta_{-m,k}(\tau) = \theta_{2k-m,k}(\tau)$, we obtain

$$(\theta_{4,12} - \theta_{8,12}) \left[(\theta_{1,12} + \theta_{11,12})^2 + (\theta_{5,12} + \theta_{7,12})^2 \right] = (\theta_{0,12} - \theta_{12,12}) (\theta_{3,12} + \theta_{9,12})^2. \quad (99)$$

At this stage, we use the following properties of the theta function [40]

$$\theta_{m,4k}(\tau) = \sum_{\ell=0}^{2k-1} \theta_{2km+16k^2\ell,16k^3}(\tau), \quad (100)$$

$$\theta_{m,k}(\tau) = \theta_{2m,4k}(\tau) + \theta_{4k-2m,4k}(\tau). \quad (101)$$

Also due to the product formula for the theta function

$$\theta_{m,k}(\tau) \theta_{m',k'}(\tau) = \sum_{\ell=1}^{k+k'} \theta_{mk'-m'k+2\ell kk'}(\tau) \theta_{m+m'+2\ell k,k+k'}(\tau), \quad (102)$$

we can obtain the useful formula

$$(\theta_{m,2k} \pm \theta_{2k-m,2k}) (\theta_{m',2k} \pm \theta_{2k-m',2k}) (\tau) = \theta_{\frac{m-m'}{2},k} \theta_{\frac{m+m'}{2},k}(\tau) \pm \theta_{k-\frac{m+m'}{2},k} \theta_{k-\frac{m-m'}{2},k}(\tau). \quad (103)$$

We multiply $(\theta_{2,12} - \theta_{10,12})$ with both sides in (99), and using the relation (103) in the following combination

$$\begin{aligned} (\theta_{2,12} - \theta_{10,12}) (\theta_{4,12} - \theta_{8,12}) &= (\theta_{1,6} - \theta_{5,6}) \theta_{3,6}, \\ (\theta_{0,12} - \theta_{12,12}) (\theta_{2,12} - \theta_{10,12}) &= \theta_{1,6}^2 - \theta_{5,6}^2, \\ (\theta_{1,12} + \theta_{11,12})^2 &= \theta_{0,6} \theta_{1,6} + \theta_{5,6} \theta_{6,6}, \\ (\theta_{5,12} + \theta_{7,12})^2 &= \theta_{0,6} \theta_{5,6} + \theta_{1,6} \theta_{6,6}, \\ (\theta_{3,12} + \theta_{9,12})^2 &= (\theta_{0,6} + \theta_{6,6}) \theta_{3,6}, \end{aligned}$$

we can prove the (99), or the original identity in (41) exactly.

Other type of the identity in (41), $\text{Ch}_{2,0}^{NS,(4)} = B_1 C_1$ without summation in the left hand side, can be easily checked only with the infinite product formula (82).

2. E_6 case

In the same method using (82) as D_4 case, we can rewrite (56) into

$$\begin{aligned} &[(\theta_{8,24} - \theta_{16,24}) (\theta_{10,24} - \theta_{14,24})] [(\theta_{5,24} + \theta_{19,24})^2] \\ &+ [(\theta_{8,24} - \theta_{16,24}) (\theta_{2,24} - \theta_{22,24})] [(\theta_{11,24} + \theta_{13,24})^2] \\ &= [(\theta_{6,24} - \theta_{18,24}) (\theta_{4,24} - \theta_{20,24})] [(\theta_{9,24} + \theta_{15,24})^2]. \end{aligned} \quad (104)$$

We use (103) for terms in each square bracket, and expand both sides. Then we can explicitly prove the equation (56). Other identity can be proved along the similar lines.

3. E_8 case

We prove (70). Using (82), we rewrite (70) into the following form.

$$(\theta_{4,60} - \theta_{56,60}) (\theta_{14,60} - \theta_{46,60}) (\theta_{16,60} - \theta_{44,60}) (\theta_{26,60} - \theta_{34,60})$$

$$\begin{aligned}
& \times \left[(\theta_{28,60} - \theta_{32,60}) (\theta_{11,60} + \theta_{49,60})^2 (\theta_{19,60} + \theta_{41,60})^2 \left((\theta_{1,60} + \theta_{59,60})^2 + (\theta_{29,60} + \theta_{31,60})^2 \right) \right. \\
& \quad \left. + (\theta_{8,60} - \theta_{52,60}) (\theta_{1,60} + \theta_{59,60})^2 (\theta_{29,60} + \theta_{31,60})^2 \left((\theta_{11,60} + \theta_{49,60})^2 + (\theta_{19,60} + \theta_{41,60})^2 \right) \right] \\
= & (\theta_{0,60} - \theta_{60,60}) (\theta_{10,60} - \theta_{50,60}) (\theta_{12,60} - \theta_{48,60}) (\theta_{18,60} - \theta_{42,60}) (\theta_{20,60} - \theta_{40,60}) \\
& \times (\theta_{3,60} + \theta_{57,60})^2 (\theta_{15,60} + \theta_{45,60})^2 (\theta_{27,60} + \theta_{33,60})^2. \tag{105}
\end{aligned}$$

Then we multiply $(\theta_{0,60} - \theta_{60,60}) (\theta_{2,60} - \theta_{58,60}) (\theta_{22,60} - \theta_{38,60})$ in both sides, and rewrite

$$\begin{aligned}
& [(\theta_{0,60} - \theta_{60,60}) (\theta_{22,60} - \theta_{38,60})] [(\theta_{2,60} - \theta_{58,60}) (\theta_{4,60} - \theta_{56,60})] \\
& \times [(\theta_{14,60} - \theta_{46,60}) (\theta_{16,60} - \theta_{44,60})] [(\theta_{26,60} - \theta_{34,60}) (\theta_{28,60} - \theta_{32,60})] \\
& \times [(\theta_{11,60} + \theta_{49,60})^2] [(\theta_{19,60} + \theta_{41,60})^2] \left([(\theta_{1,60} + \theta_{59,60})^2] + [(\theta_{29,60} + \theta_{31,60})^2] \right) \\
& + [(\theta_{0,60} - \theta_{60,60}) (\theta_{2,60} - \theta_{58,60})] [(\theta_{16,60} - \theta_{44,60}) (\theta_{22,60} - \theta_{38,60})] \\
& \times [(\theta_{4,60} - \theta_{56,60}) (\theta_{26,60} - \theta_{34,60})] [(\theta_{8,60} - \theta_{52,60}) (\theta_{14,60} - \theta_{46,60})] \tag{106} \\
& \times [(\theta_{1,60} + \theta_{59,60})^2] [(\theta_{29,60} + \theta_{31,60})^2] \left([(\theta_{11,60} + \theta_{49,60})^2] + [(\theta_{19,60} + \theta_{41,60})^2] \right) \\
= & [(\theta_{0,60} - \theta_{60,60}) (\theta_{2,60} - \theta_{58,60})] [(\theta_{0,60} - \theta_{60,60}) (\theta_{22,60} - \theta_{38,60})] \\
& \times [(\theta_{10,60} - \theta_{50,60}) (\theta_{12,60} - \theta_{48,60})] [(\theta_{18,60} - \theta_{42,60}) (\theta_{20,60} - \theta_{40,60})] \\
& \times [(\theta_{3,60} + \theta_{57,60})^2] [(\theta_{15,60} + \theta_{45,60})^2] [(\theta_{27,60} + \theta_{33,60})^2].
\end{aligned}$$

We use (103) in order to expand each square bracket. Then by expanding all the terms in a straightforward way and comparing both sides, we can check that the equation (106) holds. Thus we have obtained the complete proof of the identity (70).

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